DESIGN OF DIAMOND-SHAPED R.C. PLATES ON THE BASIS OF A NON-LINEAR PHYSICAL RELATIONSHIP

By

F. NÉMETH—ZS. GÁSPÁR

Department of Civil Engineering Mechanics, Technical University, Budapest

Received June 21, 1973

Differential equation of anisotropic plates

Design of the plate is based on a non-linear physical relationship. Loads are applied in equal increments. Within each load interval, rigidities are considered as constant so that the curve of state change is approximated by its chords. Thereby this physically non-linear problem is reduced to a set of linear ones. In computation, consideration is due to the fact that, since the curvature differs in each direction, also the bending condition stages differ, yields are of different degrees. This fact is taken into consideration by distinguishing rigidity coefficients with exchanged subscripts.

In what follows, relationships valid within a load interval — where the physical relationship is linear — will be considered. Hence, in the following formulae, stresses mean load increments within an interval. Plate equilibrium equations are:

$$q_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{yx}}{\partial y} \tag{1}$$

$$q_y = \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} \tag{2}$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = -p\,. \tag{3}$$

Moments can be calculated from the second derivatives (curvatures) of the deflection surface w, and from the stiffness coefficients as follows:

$$\begin{aligned} m_x &= -\left(D_{11} w_{xx} + D_{12} w_{yy} + 2D_{13} w_{xy}\right) \\ m_y &= -\left(D_{21} w_{xx} + D_{22} w_{yy} + 2D_{23} w_{xy}\right) \\ m_{xy} &= -\left(D_{31} w_{xx} + D_{32} w_{yy} + 2D_{33} w_{xy}\right). \end{aligned} \tag{4}$$

Substituting derivatives of (4) into (1, 2):

$$\begin{aligned} q_x &= -\,(D_{11}w_{xxx} + D_{12}w_{xyy} + 2D_{13}w_{xxy} + D_{31}w_{xxy} + D_{32}w_{yyy} + 2D_{33}w_{xyy}) \\ q_y &= -\,(D_{21}w_{xxy} + D_{22}w_{yyy} + 2D_{23}w_{xyy} + D_{31}w_{xxx} + D_{32}w_{xyy} + 2D_{33}w_{xxy})\,. \end{aligned} \tag{5}$$

Now, substituting derivatives of (5) into (3) and reducing yields the differential equation of the anisotropic plate:

$$\begin{aligned} D_{11}w_{xxxx} + 2(D_{13} + D_{31})w_{xxxy} + (D_{12} + D_{21} + 4D_{33})w_{xxyy} + \\ + 2(D_{23} + D_{32})w_{xyyy} + D_{22}w_{yyyy} = p\,. \end{aligned} \tag{6}$$

In this formula the stiffness coefficients D_{ik} are not constant but are functions of the curvatures because of the non-linear physical relationship. Introducing relative stiffness coefficients

$$\delta_{ik} = \frac{D_{ik}}{D_0} \tag{7}$$

and substituting them into (4) transforms the moment formulae as:

$$\begin{aligned} m_x &= -D_0(\delta_{11}w_{xx} + \delta_{12}w_{yy} + 2\delta_{13}w_{xy}) \\ m_y &= -D_0(\delta_{21}w_{xx} + \delta_{22}w_{yy} + 2\delta_{23}w_{xy}) \\ m_{xy} &= -D_0(\delta_{31}w_{xx} + \delta_{32}w_{yy} + 2\delta_{33}w_{xy}) \end{aligned} \tag{8}$$

D_0 being a stiffness coefficient of constant value. The differential equation will be solved numerically and to this aim it will be transformed into difference equations. Replacing derivatives by difference quotients, for a square net of mesh a, Eq. (6) yields the difference equation of anisotropic plates, symbolically written as:

Operator δ is shown in Fig. 1. $\boxed{\delta}\,\boxed{w} = \frac{pa^4}{D_0}$

Physical equations

In relationships (4) and (8) for moments and curvatures, members are assumed as seen in Fig. 2, and the non-linear relationship is expressed by an exponential function considered as working hypothesis.

According to the above, the relative rigidities can be written as:

$$\begin{aligned} \delta_{i1} &= \delta_{i10}\, e^{-\left|\frac{g_x}{g_{x0}}\right|} \qquad & i &= 1, 2, 3 \\ \delta_{i2} &= \delta_{i20}\, e^{-\left|\frac{g_y}{g_{y0}}\right|} \qquad & i &= 1, 2, 3 \\ \delta_{i3} &= \delta_{i30}\, e^{-\left|\frac{g_{xy}}{g_{xy0}}\right|} \qquad & i &= 1, 2, 3. \end{aligned} \tag{10}$$

	$+\frac{\delta_{23}}{2}$ $+\frac{\delta_{32}}{2}$	$+\delta_{22}$	$-\frac{\delta_{23}}{2}$ $-\frac{\delta_{32}}{2}$	
$+\frac{\delta_{13}}{2}$ $+\frac{\delta_{31}}{2}$	$-\delta_{13}-\delta_{31}$ $-\delta_{23}-\delta_{32}$ $+\delta_{12}+\delta_{21}+4\delta_{33}$	$-2\delta_{12}-2\delta_{21}$ $-4\delta_{22}$ $-8\delta_{33}$	$+\delta_{13}+\delta_{31}$ $+\delta_{23}+\delta_{32}$ $+\delta_{12}+\delta_{21}+4\delta_{33}$	$-\frac{\delta_{13}}{2}$ $-\frac{\delta_{31}}{2}$
$+\delta_{11}$	$-4\delta_{11}$ $-2\delta_{12}-2\delta_{21}$ $-8\delta_{33}$	$+6\delta_{11}+6\delta_{22}$ $+4\delta_{12}+4\delta_{21}$ $+16\delta_{33}$	$-4\delta_{11}$ $-2\delta_{12}-2\delta_{21}$ $-8\delta_{33}$	$+\delta_{11}$
$-\frac{\delta_{13}}{2}$ $-\frac{\delta_{31}}{2}$	$+\delta_{13}+\delta_{31}$ $+\delta_{23}+\delta_{32}$ $+\delta_{12}+\delta_{21}+4\delta_{33}$	$-2\delta_{12}-2\delta_{21}$ $-4\delta_{22}$ $-8\delta_{33}$	$-\delta_{13}-\delta_{31}$ $-\delta_{23}-\delta_{32}$ $+\delta_{12}+\delta_{21}+4\delta_{33}$	$+\frac{\delta_{13}}{2}$ $+\frac{\delta_{31}}{2}$
	$-\frac{\delta_{23}}{2}$ $-\frac{\delta_{32}}{2}$	$+\delta_{22}$	$+\frac{\delta_{23}}{2}$ $+\frac{\delta_{32}}{2}$	

operator δ

Fig. 1

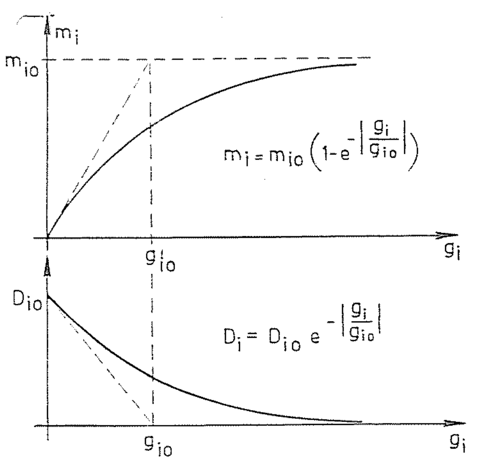

Fig. 2

In these formulae the initial relative rigidities δ_{110}, $\delta_{120} = \delta_{210}$, $\delta_{130} = \delta_{310}$, δ_{220}, $\delta_{230} = \delta_{320}$, δ_{330} are constant, and so are the relating curvature values

$$g_{x0}, g_{y0}, g_{xy0}.$$

Plate loads are considered to be one-parameter and monotonously increasing ones. Load increments will be step-wise.

Curvatures will be indicated either by g_x, g_y, g_{xy}, or by w_{xx}, w_{yy}, w_{xy}, but so that w means curvatures for one load interval each, while g means total curvatures.

Computing process

Let us compute plate deflections and moments. The acting load is uniformly distributed and applied on the plate in equal layers. The plate will be designed separately for load increments constituting the load intervals, and in this course, the curvilinear physical correlation is replaced by a rectilinear tangent each at the total curvature point. Thereby the partial differential equation of variable coefficient is reduced to a set of equations with constant coefficients.

The computation has been elaborated for the diamond-shaped r. c. plate of 45° corners, with hinged boundaries (Fig. 3). Assuming the uniform mesh network as seen in the figure ($k = 8$), and taking also point symmetry into consideration, there are 25 points where the deflection w has to be computed. Thus, the difference equations deliver a linear equation system of 25 unknowns.

Plate data

Two plates will be computed, differing only by reinforcement direction. The following data are common for both:

Span	$L = 400$ cm	
Thickness	$v = 14$ cm	
Reinforcement in direction ξ	$\varnothing$ 14/15 cm	
Reinforcement in direction η	$\varnothing$ 12/30 cm	
Ultimate moments	$\overline{m}_\xi = 2270$ cmkp/cm	
	$\overline{m}_\eta = 720$ cmkp/cm	
Moduli of elasticity	$E_b = 320$ Mp/cm^2	(concrete)
	$E_a = 2100$ Mp/cm^2	(steel)
Moments of inertia	$I_\xi = 243$ cm^4/cm	
	$I_\eta = 231$ cm^4/cm	
Initial stiffness	$D_0 = D_{\xi 0} = 7.85 \cdot 10^7$ cm^{2}kp/cm	
Coefficient of transversal expansion	$\nu = 0.1$	

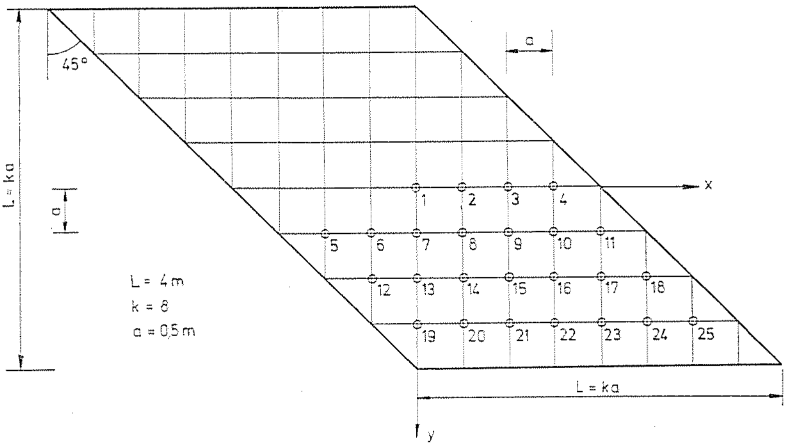

Fig. 3

Both reinforcements are orthogonal, that of type **A** being in direction x,y and of type **B** rotated at an angle of 45°, the principal reinforcement being orthogonal to the longer side. Ultimate moments of the plates in co-ordinate direction, stiffnesses along the reinforcement and in the direction of co-ordinates are compiled in Table 1. These latter will be computed by LEKHNITZKY's formulae [1]. Tables indicate also the relating curvatures. Load increments $\Delta p = 0.025$ kp/cm^2 = 250 kp/m^2. Two kinds of analyses have been incorporated into the computer program. The one compares the maximum curvature to the permissible curvature

$$g_e = 0.78 \cdot 10^{-4}\ \mathrm{cm}^{-1}\,.$$

This kind of analysis corresponds to crack control. The other analysis program relates maximum deflection to the permissible one:

$$w_e = 2\ \mathrm{cm}\,.$$

These permissible values have been assumed arbitrarily.

Outputs

The computation applied a program written in ALGOL language for the ODRA—1204 computer of the Faculty of Civil Engineering of the Technical University, Budapest. Total deflections and moments in every load interval were obtained as outputs. Mid-plate deflection is shown in Fig. 4, together with points belonging to various ultimate conditions. It is interesting to see the differential behaviour of plates type **A** and **B**, differing only by the reinforcement direction.

Table 1

	Type A	Type B	Dimension
m_{x0}	2270	1495	cm kp/cm
m_{y0}	720	1495	,,
m_{xy0}	0	775	,,
$D_{\xi 0}$	$7.85 \cdot 10^7$	$7.85 \cdot 10^7$	cm²kp/cm
$D_{\eta 0}$	$7.48 \cdot 10^7$	$7.48 \cdot 10^7$	,,
$D_{\xi\eta 0}$	$3.83 \cdot 10^7$	$3.83 \cdot 10^7$	,,
D_{110}	$7.85 \cdot 10^7$	$5.75 \cdot 10^7$	,,
$D_{120} = D_{210}$	0	$2.67 \cdot 10^7$	,,
$D_{130} = D_{310}$	0	$-0.092 \cdot 10^7$	,,
D_{220}	$7.48 \cdot 10^7$	$5.75 \cdot 10^7$	,,
$D_{230} = D_{320}$	0	$-0.092 \cdot 10^7$	,,
D_{330}	$3.83 \cdot 10^7$	$4.97 \cdot 10^7$	,,
δ_{110}	1	0.743	—
$\delta_{120} = \delta_{210}$	0	0.345	—
$\delta_{130} = \delta_{310}$	0	-0.012	—
δ_{220}	0.953	0.743	—
$\delta_{230} = \delta_{320}$	0	-0.012	—
δ_{330}	0.487	0.642	—
g_{x0}	$0.289 \cdot 10^{-4}$	$0.260 \cdot 10^{-4}$	cm⁻¹
g_{y0}	$0.096 \cdot 10^{-4}$	$0.260 \cdot 10^{-4}$	,,
g_{xy0}	0	$0.156 \cdot 10^{-4}$	,,

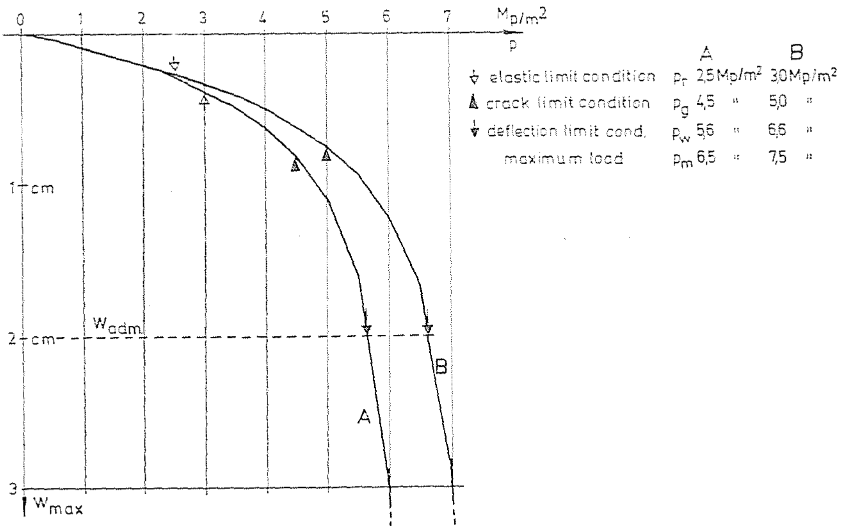

Fig. 4

Magnitudes and directions of principal moments at all points of load increments have been determined; their values in the first and last load interval are shown in Figs 5 to 8. Tending to the ultimate load, the directions of the principal moments assume ever more that of the reinforcement.

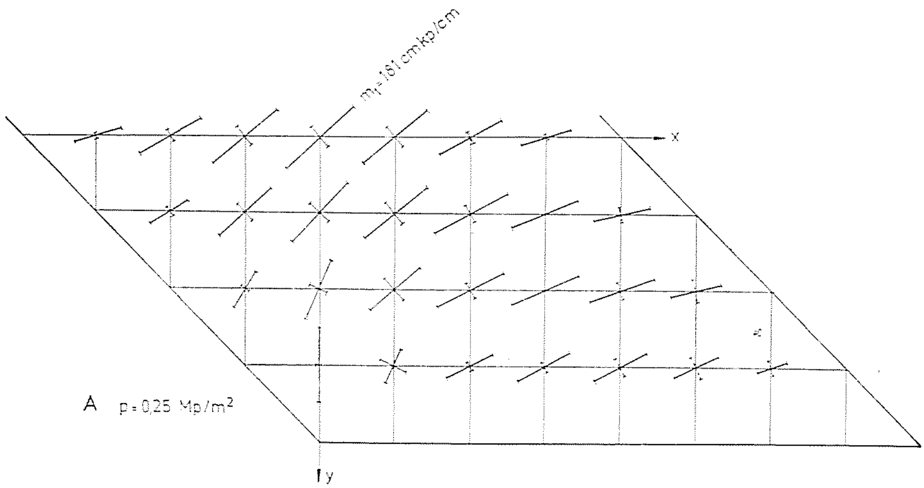

Fig. 5. m_1, m_2 principal moments, ——— positive; - - - - - - - - negative

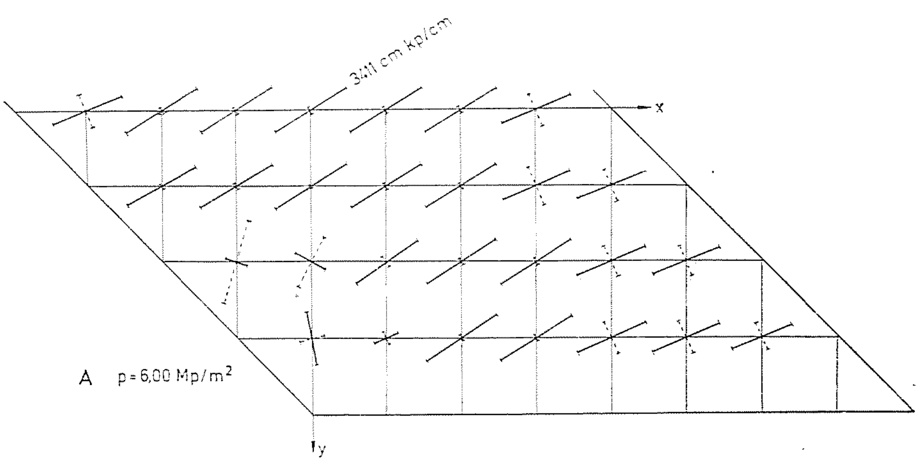

Fig. 6

Summary

The design of r. c. plates for physical non-linearity has been presented. The relevant differential equation of anisotropic plates is also valid for non-linear physical relationships. Variation of stiffness coefficients has been described by means of exponential functions considered as work hypothesis. Loads have been applied in uniform increments. Within each load

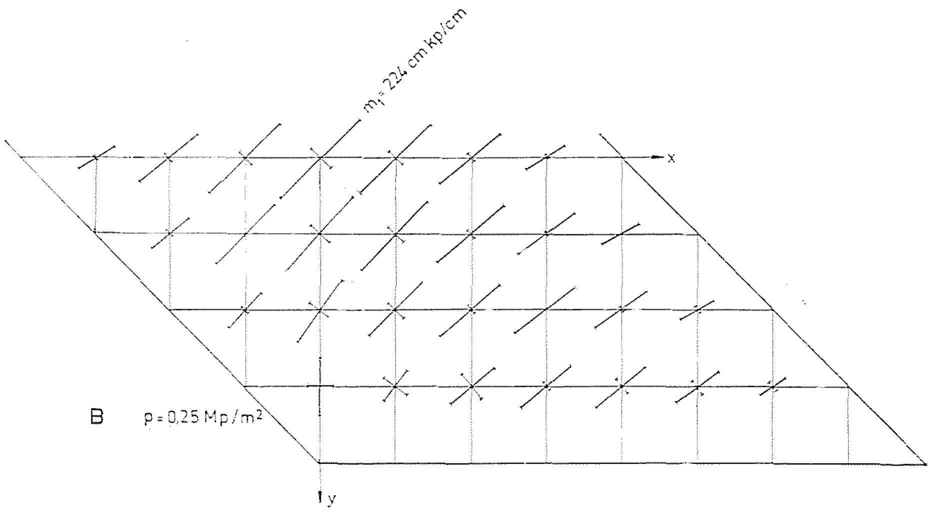

Fig. 7

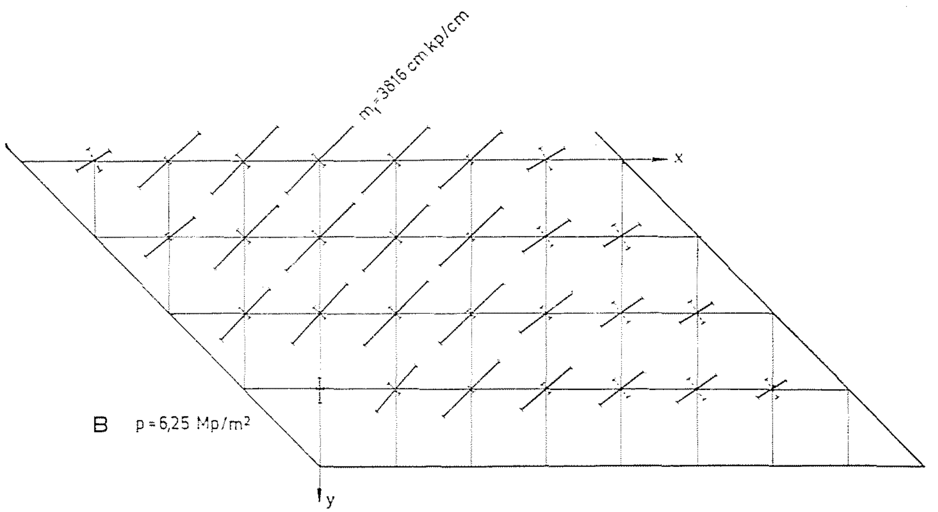

Fig. 8

interval, rigidities have been considered as constant. Plate deflections and moments have been computed by the method of finite differences. The computation has been made for two r. c. plates differing only by the reinforcement direction. Plate moments and response have largely been affected by the reinforcement.

References

1. LEKHNITZKY, S. G.: Anisotropnie plastinki. Ogis. Gostekhisdat. 1947.

Ass. Prof. Dr. Ferenc NÉMETH
Res. worker Dr. Zsolt GÁSPÁR
1111 Budapest, Műegyetem rkp. 3, Hungary